Malaysian Journal of Mathematical Sciences 4(2): 183-194 (2010)

A Pursuit-Evasion Differential Game with Many Pursuers and One Evader

[1]Gafurjan Ibragimov and [2]Noor Aidawati Hussin
[1]Institute for Mathematical Research and Department of Mathematics, Faculty of Science, Universiti Putra Malaysia, 43400 UPM Serdang, Selangor, Malaysia
[2]Department of Mathematics, Faculty of Science, Universiti Putra Malaysia, 43400 Serdang, Selangor, Malaysia
E-mail: gafur@science.upm.edu.my and noor.aidawati@treasury.gov.my

ABSTRACT

We study a pursuit-evasion differential game of many players with geometric constraints being imposed on the control parameters of players. Game is described by an infinite system of differential equations of second order in Hilbert space. Duration of the game is fixed. Payoff is the infimum of the distances between the evader and pursuers when the game is terminated. The pursuers' goal is to minimize the payoff, and the evader's goal is to maximize it. A condition to find the value of the differential game is obtained. Optimal strategies of players are also constructed.

Keywords: Differential game, pursuit, evasion, control, strategy, value of the game.

INTRODUCTION

Many works have been devoted to differential games, such as those by Isaacs, (1965), Pontryagin, (1988), Friedman, (1971), Krasovskii and Subbotin, (1988) and etc. Constructing the players' optimal strategies and finding the value of the game are of specific interest in studying differential games.

Pursuit-evasion differential games involving several objects with simple motions get the attention of many authors. For example, Ivanov and Ledyaev, (1981) studied simple motion differential game of several players with geometric constraints. They obtained sufficient conditions to find optimal pursuit time in $\mathbb{R}^n$.

Levchenkov and Pashkov, (1985) investigated differential game of optimal approach of two identical inertial pursuers to a noninertial evader on a fixed time interval. Control parameters were subject to geometric constraints. They constructed the value function of the game and used necessary and sufficient conditions which a function must satisfy to be the value function (Subbotin and Chentsov, (1981)).

In Ibragimov, (2005), a differential game with many pursuers and geometric constraints was investigated in the Hilbert space l_2.

In the present paper, we study a pursuit-evasion differential game with many pursuers and one evader with geometric constraints on the controls of players described by equations

$$\ddot{x}_i = u_i, \quad x_i(0) = x_i^0, \quad \dot{x}_i(0) = x_i^1, \quad \| u_i \| \le \rho_i, \quad i = 1, 2, ..., \tag{1}$$

$$\ddot{y} = v, \quad y(0) = y^0, \quad \dot{y}(0) = y^1, \quad \| v \| \le \sigma, \tag{2}$$

where $x_i, x_i^0, x_i^1, y, y^0, y^1, u_i, v \in l_2$, u_i are control parameters of the pursuers and v is that of the evader, ρ_i and σ are given positive numbers, and $\| v \| = \left(\sum_{i=1}^{\infty} v_i^2 \right)^{1/2}$.

The duration of the game, denoted θ, is fixed. The payoff function is the infimum of the distances between the evader and the pursuers at θ:

$$\gamma(\theta) = \inf_{1 \le i \le \theta} \| x_i(\theta) - y(\theta) \|. \tag{3}$$

The pursuers' goal is to minimize the payoff, and the evader's goal is to maximize it.

Denote $H(0, \rho)$ as the ball in l_2 of the radius ρ with the center at the origin.

Definition 1. A function $u_i(t), 0 \le t \le \theta$, $u_i : [0, \theta] \to H(0, \rho_i)$, with measurable coordinates $u_{ik}(t), k = 1, 2, ...$ is called an admissible control of the pursuer x_i.

Definition 2. A function $v(t), 0 \le t \le \theta$, $v : [0, \theta] \to H(0, \sigma)$, with measurable coordinates $v_i(t)$ is called an admissible control of the evader y.

Once the players' admissible controls $u_i(t)$ and $v(t),\ 0 \le t \le \theta,$ are chosen, the corresponding motions $x_i(t) = (x_{i1}(t),..., x_{ik}(t),...)$ and $y(t) = (y_1(t),..., y_k(t),...)$ of the players are defined as

$$x_{ik}(t) = x_{ik}^0 + tx_{ik}^1 + \int_0^t \int_0^s u_{ik}(r)drds, \quad y_k(t) = y_k^0 + ty_k^1 + \int_0^t \int_0^s v_k(r)drds\ .$$

One can readily see that $x_i(\cdot), y(\cdot) \in C(0,\theta,l_2)$, where $C(0,\theta,l_2)$ is the space of functions

$$f(t) = (f_1(t),..., f_k(t),...) \in l_2,\ t \ge 0,$$

subject to the conditions

(1) $f_k(t),\ 0 \le t \le \theta,\ k = 1,2,...,$ are absolutely continuous functions;
(2) $f(t),\ 0 \le t \le \theta,$ is continuous in the norm of l_2.

Definition 3. A function $U_i(x_i, y, v), U_i : l_2 \times l_2 \times H(0,\sigma) \to H(0,\rho_i)$, such that the system

$$\ddot{x}_i = U_i(x_i, y, v),\ x_i(0) = x_i^0,\ \dot{x}_i(0) = x_i^1,$$
$$\ddot{y} = v,\ y(0) = y^0,\ \dot{y}(0) = y^1,$$

has a unique solution $(x_i(\cdot),\ y(\cdot)) \in C(0,\theta,l_2)$ for an arbitrary admissible control $v(t)$ of the evader is called a strategy of the pursuer. The strategy U_i is said to be admissible if each control generated by this strategy is admissible.

Definition 4. Strategies U_{i0} of the pursuers are said to be optimal if

$$\inf_{U_1,...,U_m,...} \Gamma_1(U_1,...,U_m,...) = \Gamma_1(U_{10},...,U_{m0},...)\ ,$$

where

$$\Gamma_1(U_1,...,U_m,...) = \sup_{v(\cdot)} \inf_i \| x_i(\theta) - y(\theta) \|,$$

U_i are admissible strategies of the pursuers and $v(\cdot)$ is an admissible control of the evader.

Definition 5. A function $V(y, x_1, ..., x_m, ...), V: l_2 \times l_2 \times ... \times l_2 \times ... \to H(0, \sigma)$, such that the following countable system of equations

$$\ddot{x}_k = u_k,\ x_k(0) = x_k^0,\ \dot{x}_k(0) = x_k^1,\ k = 1, 2, ...,$$
$$\ddot{y} = V(y, x_1, ..., x_m, ...),\ y(0) = y^0,\ \dot{y}(0) = y^1,$$

has a unique solution $(y(\cdot), x_1(\cdot), ..., x_i(\cdot), ...)$, $x_i(\cdot), y(\cdot) \in C(0, \theta, l_2)$ for arbitrary admissible controls $u_i = u_i(t), 0 \le t \le \theta,$ of the pursuers is called a strategy of the evader. If each control formed by the strategy V is admissible, then the strategy V itself is said to be admissible.

Definition 6. A strategy V_0 of the evader is said to be optimal if $\sup_V \Gamma_2(V) = \Gamma_2(V_0),$ where

$$\Gamma_2(V) = \inf_{u_1(\cdot), ..., u_m(\cdot), ...} \inf_i \| x_i(\theta) - y(\theta) \|,$$

$u_i(\cdot)$ are admissible controls of the pursuers, V is an admissible strategy of the evader.

If $\Gamma_1(U_{10}, ..., U_{m0}, ...) = \Gamma_2(V_0) = \gamma$,

then we say that the game has the value γ (Subbotin and Chentsov, (1981)). The problem is to find optimal strategies $U_{10}, ..., U_{m0}, ...$ of the pursuers and the optimal strategy V_0 of the evader, and the value of the game.

Differential game (1)-(3) can be reduced to an equivalent game described by the equations (Ibragimov and Mehdi, (2009)).

$$\dot{x}_i = (\theta - t)u_i,\ x_i(0) = x_{i0},\ \| u_i \| \le \rho_i,\ i = 1, 2, 3, ... \tag{4}$$

$$\dot{y} = (\theta - t)v, \quad y(0) = y_0,\ \| v \| \le \sigma, \tag{5}$$

with the payoff (3) where x_{i0} and y_0 are some points.

MAIN RESULTS

Let

$$\gamma = \inf\left\{ l \ge 0 \middle| \quad H\left(y_0, \frac{1}{2}\sigma\theta^2 \right) \subset \bigcup_{i\in I} H\left(x_{i0}, \frac{1}{2}\rho_i\theta^2 + l \right) \right\}, \quad I = \{1,2,....\}.$$

Theorem 2.1. If there exists a nonzero vector $p \in l_2$ such that $(y_0 - x_{i0}, p) \ge 0$ and $\sigma \le \rho_i + \frac{2\gamma}{\theta^2}$ for all $i \in I$, then the number γ is the value of the game (1)-(3).

Proof.

1. *Fictitious pursuers.* To construct strategies of the pursuers we introduce fictitious pursuers z_i, whose motions are described by the equations

$$\dot{z}_i = (\theta - t)\, w_i^{\varepsilon}, \quad z_i(0) = x_{i0}, \quad \left\| w_i^{\varepsilon} \right\| \le \rho_i + \frac{2\gamma}{\theta^2} + \frac{2\varepsilon}{k_i\theta^2} = \rho_i(\varepsilon),$$

where $\varepsilon < 1,$ is an arbitrary positive number and $k_i = \max\{1, \rho_i\}$. The domain of attainability of the fictitious pursuer z_i from the initial position x_{i0} up to the time θ is the ball

$$H\left(x_{i0}, \frac{1}{2}\rho_i(\varepsilon)\theta^2 \right) = H\left(x_{i0}, \frac{1}{2}\rho_i\theta^2 + \gamma + \frac{\varepsilon}{k_i} \right).$$

We define the strategies of the fictitious pursuers z_i on $[0, \theta]$ as follows:

$$w_i^{\varepsilon}(t) = v(t) - (v(t), e_i)e_i + e_i\sqrt{\rho_i^2(\varepsilon) - \sigma^2 + (v(t), e_i)^2}, \quad 0 \le t < \tau_i,$$

$$w_i^{\varepsilon}(t) = v(t), \qquad \tau_i \le t \le \theta, \tag{6}$$

where

$$e_i = \begin{cases} \dfrac{y_0 - x_{i0}}{\left\| y_0 - x_{i0} \right\|}, & x_{i0} \ne y_0, \\ 0, & x_{i0} = y_0, \end{cases}$$

and τ_i , $0 \le \tau_i \le \theta$, is defined as the first time at which $z_i(\tau_i) = y(\tau_i)$. Note that such a time τ_i may not be exist.

2. *Construction of the strategies of the real pursuers.* We define the strategies of the real pursuers x_i as follows:

$$u_i(t) = \frac{\rho_i}{\bar{\rho}_i} w_i(t), \ \ \bar{\rho}_i = \rho_i(0) = \rho_i + \frac{2\gamma}{\theta^2}, \tag{7}$$

where

$$w_i(t) = v(t) - (v(t), e_i)e_i + e_i\sqrt{\bar{\rho}_i^2 - \sigma^2 + (v(t), e_i)^2}, \ 0 \le t \le \tau_i,$$
$$w_i(t) = v(t), \qquad \tau_i \le t \le \theta.$$

It is clear that $\| w_i(t) \| \le \bar{\rho}_i$.

3. *We show that γ is the guaranteed value for the pursuers*. We prove that if the pursuers use the strategy (7), then $\sup_{v(\bullet)} \inf_{i\in I} \left\| y(\theta) - x_i(\theta) \right\| \le \gamma$,

where $v(\cdot)$ is a control of the evader. According to the definition of θ we have

$$H\left(y_0, \frac{1}{2}\sigma\theta^2 \right) \subset \bigcup_{i\in I} H\left(x_{i0}, \frac{1}{2}\rho_i\theta^2 + \gamma + \frac{\varepsilon}{k_i} \right)$$

and by the hypothesis of the theorem

$$(y_o - x_{i0}, p) \ge 0, i \in I,$$

then, Assertion 4 (Ibragimov, (2005)) implies

$$H\left(y_0, \frac{1}{2}\sigma\theta^2 \right) \subset \bigcup_{i\in I} X_i^{\varepsilon}, \tag{8}$$

where

$$X_i^\varepsilon = \left\{ z \middle| \; 2\left(y_0 - x_{i0}, z\right) \le \left(\frac{1}{2}\rho_i\theta^2 + \gamma + \frac{\varepsilon}{k_i}\right)^2 - \left(\frac{1}{2}\sigma\theta^2\right)^2 + \|y_0\|^2 - \|x_{i0}\|^2 \right\}, \;\; x_{i0} \ne y_0,$$

$$X_i^\varepsilon = \left\{ z \middle| \; \left(z - x_{i0}, e_i\right) \le \frac{1}{2}\rho_i\theta^2 + \gamma + \frac{\varepsilon}{k_i}, \quad x_{i0} = y_0 \right\}.$$

Hence, $y(\theta) \in X_s^\varepsilon$ for some $s \in I$.

The following Lemma can be proved in much the same way as the Lemma in Ibragimov, (2005).

Lemma 2.1. *If* $\sigma \le \rho_i(\varepsilon)$ *and* $y(\theta) \in X_i^\varepsilon$ *, then the fictitious pursuer's strategy* (6) *ensures that* $z_i(\theta) = y(\theta)$.

According to this Lemma we obtain that if the fictitious pursuers apply the strategy (6), then $z_s(\theta) = y(\theta)$ and hence

$$\|y(\theta) - x_s(\theta)\| = \|z_s(\theta) - x_s(\theta)\|$$

$$= \left\| x_{s0} + \int_0^\theta (\theta - t) w_s^\varepsilon(t)dt - x_{s0} - \int_0^\theta (\theta - t) u_i(t)dt \right\|$$

$$= \left\| \int_0^\theta (\theta - t) \left(w_s^\varepsilon(t) - w_s(t) + w_s(t) - \frac{\rho_s}{\bar{\rho}_s} w_s(t) \right) dt \right\|$$

$$\le \int_0^\theta (\theta - t) \| w_s^\varepsilon(t) - w_s(t) \| \, dt + \int_0^\theta (\theta - t) \| w_s(t) - \frac{\rho_s}{\bar{\rho}_s} w_s(t) \| \, dt. \qquad (9)$$

We show that

$$\lim_{\varepsilon \to 0} \sup_{i \in I} \int_0^\theta (\theta - t) \left\| w_i^\varepsilon(t) - w_i(t) \right\| dt = 0. \qquad (10)$$

If $x_{i0} = y_0$, then by construction $w_i^\varepsilon(t) = w_i(t) = v(t)$, so

$$\int_0^\theta (\theta - t)\left\| w_i^\varepsilon(t) - w_i(t)\right\| dt = 0 .$$

If $x_{i0} \neq y_0$, then we obtain the following equation:

$$\int_0^\theta (\theta - t)\left\| w_i^\varepsilon(t) - w_i(t)\right\| dt$$

$$= \int_0^{\tau_i} (\theta - t)\left(\sqrt{\rho_i^2(\varepsilon) - \sigma^2 + (v(t), e)^2} - \sqrt{\bar{\rho}_i^{\,2} - \sigma^2 + (v(t), e)^2} \right) dt . \tag{11}$$

Next, we'll consider the following function:

$$f(\xi) = \sqrt{\rho_i^2(\varepsilon) - \sigma^2 + \xi^2} - \sqrt{\bar{\rho}_i^2 - \sigma^2 + \xi^2}, \qquad 0 \le \xi \le \sigma.$$

It is not difficult to show that this function takes its maximum at $\xi = 0$. Therefore from (11) we obtain

$$\int_0^\theta (\theta - t)\left\| w_i^\varepsilon(t) - w_i(t)\right\| dt$$

$$\le \int_0^\theta (\theta - t)\left(\sqrt{\rho_i^2(\varepsilon) - \sigma^2} - \sqrt{\bar{\rho}_i^{\,2} - \sigma^2} \right) dt$$

$$= \frac{1}{2}\theta^2 \left(\sqrt{\rho_i^2(\varepsilon) - \sigma^2} - \sqrt{\bar{\rho}_i^{\,2} - \sigma^2} \right)$$

$$\le \frac{1}{2}\theta^2 \sqrt{\rho_i^2(\varepsilon) - \bar{\rho}_i^{\,2}} . \tag{12}$$

However,

$$\rho_i^2(\varepsilon) - \bar{\rho}_i^2 = \frac{2\varepsilon}{k_i \theta^2}\left(2\rho_i + \frac{4\gamma}{\theta^2} + \frac{2\varepsilon}{k_i \theta^2} \right)$$

$$= \varepsilon \left(\frac{\rho_i}{k_i} \cdot \frac{4}{\theta^2} + \frac{8\gamma}{k_i \theta^4} + \frac{4\varepsilon}{k_i^2 \theta^4} \right) \le \varepsilon \left(\frac{4}{\theta^2} + \frac{8\gamma}{\theta^4} + \frac{4\varepsilon}{\theta^4} \right),$$

since $k_i = \max\{\rho_i, 1\}$. Then according to (12)

$$\int_0^\theta (\theta - t)\left\| w_i^\varepsilon(t) - w_i(t)\right\| dt \le \sqrt{\varepsilon\left(\frac{4}{\theta^2} + \frac{8\gamma}{\theta^4} + \frac{4\varepsilon}{\theta^4}\right)} \cdot \frac{\theta^2}{2} \le K\sqrt{\varepsilon},$$

where $K = \sqrt{\frac{4}{\theta^2} + \frac{8\gamma}{\theta^4} + \frac{4}{\theta^4}} \cdot \frac{\theta^2}{2}$. Hence (10) is true.

Since $\| w_i(t) \| \le \bar{\rho}_i$ for the second term in (9) we have

$$\int_0^\theta (\theta - t)\left\| w_s(t) - \frac{\rho}{\bar{\rho}} w_s(t)\right\| dt = \int_0^\theta (\theta - t)\left\| w_s(t)\right\| \left| 1 - \frac{\rho_s}{\bar{\rho}_s}\right| dt$$

$$\le \bar{\rho}_s \cdot \frac{2\gamma}{\bar{\rho}_s \theta^2} \int_0^\theta (\theta - t)\, dt = \gamma.$$

This means

$$\left\| y(\theta) - x_s(\theta)\right\| \le \gamma + K\sqrt{\varepsilon}\ .$$

Letting $\varepsilon \to 0$ we have

$$\left\| y(\theta) - x_s(\theta)\right\| \le \gamma.$$

4. *We show that γ is the guaranteed value for the evader.*

All that remains for us to do is to show that there exists a strategy of the evader such that

$$\inf_{u_1(\cdot),\ldots,u_m(\cdot),\ldots}\ \inf_{i \in I} \left\| y(\theta) - x_i(\theta)\right\| \ge \gamma,$$

where $u_1(\cdot), u_2(\cdot), \ldots$ are controls of the pursuers.

If $\gamma = 0,$ then any strategy of the evader, in particular, $v(t) = 0,\ 0 \le t \le \theta,$ guarantees $\gamma = 0.$ Let $\gamma > 0$. Then by the definition of γ for any $\varepsilon > 0$, of course, $\varepsilon < \gamma,$

$$H\left(y_0, \frac{1}{2}\sigma\theta^2\right) \not\subset \bigcup_{i=1}^{\infty} H\left(x_{i0},\ \frac{1}{2}\rho_i\theta^2 + \gamma - \varepsilon\right).$$

Then according to Assertion 5 (Ibragimov, (2005)), there exists a point

$$\overline{y} \in S\left(y_0, \frac{1}{2}\sigma\theta^2\right)$$

such that

$$\left\|\overline{y} - x_{i0}\right\| \geq \frac{1}{2}\rho_i\theta^2 + \gamma,\ i = 1, 2, ...,$$

where $S(y, r)$ denotes the sphere of the radius r and with the center at y.
Hence,

$$\left\|\overline{y} - x_i(\theta)\right\| \geq \left\|\overline{y} - x_{i0}\right\| - \left\|x_i(\theta) - x_{i0}\right\|$$

$$\geq \frac{1}{2}\rho_i\theta^2 + \gamma - \frac{1}{2}\rho_i\theta^2 = \gamma$$

for all $i = 1, 2, ...$.

We set

$$v(t) = \frac{2(\overline{y} - y_0)}{\theta^2}, \quad 0 \leq t \leq \theta.$$

Then

$$y(\theta) = y_0 + \int_0^{\theta} (\theta - t) \cdot \frac{2}{\theta^2}(\overline{y} - y_0)\, dt = y_0 + \overline{y} - y_0 = \overline{y},$$

and hence

$$\left\|y(\theta) - x_i(\theta)\right\| = \left\|\overline{y} - x_i(\theta)\right\| \geq \gamma,\ i = 1, 2, ...$$

Thus,

$$\inf_{i \in I} \left\|y(\theta) - x_i(\theta)\right\| \geq \gamma.$$

This completes the proof of the theorem.

CONCLUSION

A pursuit-evasion differential game of fixed duration with countably many pursuers has been studied here. Control functions satisfy geometric constraints. Under certain conditions, the value of the game has been found and the optimal strategies of players have been constructed. It should be noted that the condition given by the assumption is relevant. If this condition doesn't hold, then, in general case, we don't have a solution for the pursuit-evasion problem even in a finite dimensional space with finite number of pursuers. The present work can be extended by considering higher order differential equations.

ACKNOWLEDGEMENTS

This research was supported by the National Fundamental Research Grant Scheme (FRGS) of Malaysia, No. 05-10-07-376FR.

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